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We report noise-induced threshold shifts and phase diagrams in electroconvections (ECs) in a nematic liquid crystal by controlling the frequency band (i.e., cutoff frequency $f_c$) of noise. A crucial relationship between the internal characteristic time (i.e., charge relaxation time $\tau_C$) of the EC system and the correlation time $\tau_N$ of the external colored noise is found, which determines the role of noise in the nonequilibrium EC system. Modified numerical results using the relationship are quantitatively compared with the present experimental results. From the relationship, two types of phase diagrams can be classified in white-like and colored noises.

Interesting features of nonequilibrium phenomena in external noise have been addressed in a variety of research fields.\textsuperscript{1-3} For instance, stochastic resonance\textsuperscript{4} and noise-induced order (or phase transitions)\textsuperscript{5} are well-known as nontrivial noise phenomena. These are somewhat counterintuitive phenomena, because noise usually plays a role in negating desirable effects on controlling systems. So far, in particular, the impact of multiplicative noise in spatially extended systems has been extensively investigated to understand pattern formations in nonlinear dissipative systems.\textsuperscript{6-13} The noise can modify the thresholds and phase diagrams of dissipative structures. Moreover, colored noise has been noticed in various real application fields (having short characteristic times comparable to noise correlation time $\tau_N$) such as dye-laser statistics,\textsuperscript{12} sensory neurons,\textsuperscript{13} and population dynamics.\textsuperscript{14}

An ac-driven electroconvection (EC) system in nematic liquid crystals (NLCs) has been intensively investigated to understand the impact of noise $[\xi(t)]$, for it has a lot of merits: well-established theories,\textsuperscript{6,7} direct electro-optical measurements for threshold shift and structure change, finely adjustable (electric) noise source, controllable system times, etc.\textsuperscript{9-11} Even though the present approach is limited to ECs, it shows a crucial evidence of the role of colored noise in nonlinear dynamical systems.

In earlier studies in ECs, white noise was used both in theories\textsuperscript{5,7} and in experiments\textsuperscript{6,7} because of its practical convenience. In white noise-limit ($\tau_N \to 0$) characterized by a single parameter, noise intensity $V_N$, the threshold ($V_c$) of a primarily occurred EC [i.e., the so-called Williams domain (WD)] was theoretically described by the following equation:\textsuperscript{6,7}

\[
V_c^2 = V_{c0}^2 + bV_N^2 \quad (\text{for } b > 0).
\]  

Here, $V_{c0}$ [$= V_{c0}(0)$] indicates a threshold voltage of ac frequency $\omega$-dependent WDS for $V_N = 0$. The slope $b$ [$= b(\omega)$] stands for a sensitivity of the WDS to noise [see Eq. (3) below]. Such a threshold shift can be intuitively understood by considering random oscillations by noise; accumulation of charge due to Carr–Helfrich (CH) mechanism\textsuperscript{15,16} is reduced by the random oscillations, and therefore larger voltage is required for the onset of EC (i.e., $b > 0$). In this case, white noise plays a role of stabilization for WDS and then $V_c$ monotonically increases with increase of $V_N$.\textsuperscript{6,7}

However, it has been known that Eq. (1) is not valid for colored noise.\textsuperscript{9-11} Depending on a characteristic time (i.e., $\tau_N$) of colored noise, a destabilization effect for ECs (i.e., $b < 0$) has been appeared.\textsuperscript{8-11} To fully understand the noise-induced threshold, therefore, we should deal with colored noise ($\tau_N \neq 0$) characterized by $V_N$ and the additional parameter $\tau_N$. Here, the typical Ornstein-Uhlenbeck noise characterized with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t + \tau) \rangle = \langle \xi(0) \rangle \exp(-|\tau|/\tau_N)$ is considered. In the CH mechanism for ECs, there exist two (typical) internal characteristic times: the director relaxation time $\tau_d = \gamma_1 d^2/K_{33}$ ($\sim 10^{-1}$ s) and charge relaxation time $\tau_c = \tau_0 \varepsilon_1/\varepsilon_0$ ($\sim 10^{-3} - 10^{-4}$ s). Here, $\gamma_1$, $K_{33}$, $\varepsilon_1$, and $\varepsilon_0$ indicate a kind of viscosity, elasticity, (parallel) dielectric constant, and conductivity of NLCs, respectively.\textsuperscript{15,16} In particular, if these timescales are comparable to $\tau_N$, the situation is quite different from that of the white noise-limit ($\tau_N = 0$). For colored noise, the impact of noise to the threshold should be examined in terms of these system timescales. For colored noise, we suggest a modification of the pre-factor $b$ in Eq. (1) that can be rewritten as a function of $\tau_d$ and $\tau_N$:

\[
b_{\text{mod}} = b \left( 1 - \frac{\tau_N \tau_d}{\tau_d - \tau_N} \right) \quad (\text{for } b > 0). 
\]  

Justifiably, if $\tau_N \to 0$ (i.e., white noise), $b_{\text{mod}} \to b$. From this modification, noise-induced threshold shifts and pattern changes in the EC system can be better understood; previous studies fail to explain the appearance of $b < 0$ because of not considering such characteristic times.\textsuperscript{9-11}

In this paper, we focus our attention on the sensitivity $b_{\text{mod}}$ and the phase diagram by controlling $\tau_d$ and $\tau_N$. In particular, a quantitative comparison between numerical $b_{\text{mod}}$ and experimental results is presented.

A sinusoidal ac field $E(t) = \sqrt{2}V \sin(\omega t)/d$ was applied across a thin slab [$d = 50 \mu$m and a lateral (active) size $S = 1 \times 1 \text{cm}^2$] of an NLC [p-methoxybenzyldene-p′-n-butylaniline (MBBA)] sandwiched between two parallel and transparent electrodes (indium tin oxide). The director of the NLC was planarly aligned in the sample cell. To investigate the impact of noise on ECs, a Gaussian-type electro noise $\xi(t)$ was superimposed to the EC system sustained at a deterministic voltage ($V > V_{c0}$); consequently, a fluctuating sinusoidal field was applied across the NLC cell. Cutoff frequency $f_c$-dependent colored noise for which is
generated by the low-pass filters of a synthesizer (Hioki, 7075) was used. The correlation time \(T_N = \frac{1}{\omega N^2} = \left(2\pi f_c\right)^{-1}\) and intensity \(V_N = \frac{1}{d^2} = \frac{1}{d^2} \frac{1}{d^2}\) of noise were controlled. All measurements were performed at a stable temperature \((T = 25 \pm 0.2^\circ C)\) using an electro-thermal control system (TH-99, Japan Hightech). The details of our experiment were described in our previous papers.\(^{10}\)

First, by controlling noise intensity \(V_N\) and cutoff frequency \(f_c\), we carefully examined the onset of an EC [i.e., WD at a fixed ac frequency \(\omega/2\pi = 30Hz\)]\(^{6,11}\) and compared with the previous theory.\(^{6,7}\) For a quantitative comparison, we used \(b = b(\omega) > 0\) of Kawakubo et al.,\(^7\)

\[
b = \frac{1 + \omega^2 \tau_r^2}{\xi^2 - (1 + \omega^2 \tau_r^2)}. \quad (3)
\]

Here, \(\xi^2\) indicates a dimensionless coefficient determined by material constants of NLCs (usually, \(1.5 < \xi^2 < 4\) for MBBA;\(^{15}\)) for the present cell, \(b = 0.49\) for \(\xi^2 = 3.1\) and \(\tau_r = 9.1 \times 10^{-5}\) s. In Fig. 1, a calculated \(V_c(V_N)\) line [Eq. (1)] shows a good agreement with the measured \(V_c\) for quasi-white noise \((f_c \sim 50kHz)\).\(^{6,7,10,11}\) However, for sufficiently colored noise \((f_c < 5kHz)\) the theory is completely inapplicable (i.e., \(b < 0\)).

Next, we examined Eq. (2) to achieve \(b_{mod} < 0\) (as well as \(b_{mod} > 0\)) for colored noise \((\tau_N \neq 0)\). Similarly to resonance phenomena, we considered a relationship between an (external) timescale of noise and an (internal) timescale of the system. Calling into account \(\tau_d \sim 10^{-1}\) s in the present cell) much longer than \(\tau_r\) (\(\ll \tau_d\)), a relationship between \(\tau_N\) and \(\tau_r\) is a candidate of a first approximation to the modification. A relationship was experimentally determined in Fig. 2;

\[
\tau_N = \frac{h^{-1}}{-1 + h^{-1}} \tau_r^m, \quad (4)
\]

at which \(h = 8.1 \times 10^{-3}\) and \(m = 1.6 \pm 0.1\) (practically, several cells having different \(\tau_r\) were used). Equation (4) indicates no shift of the WD threshold \(V_c\) in the noise (i.e., \(b = 0\)).

To check the modified Eq. (2), the dependence of \(b_{mod}\) on \(f_c [1/(2\pi \tau_N)]\) was calculated. As shown in Fig. 3, Eq. (2) is considerably well fitted with experimental data for wide colored noise region \((f_c \sim 2kHz)\), and it is recovered to Eq. (3) for white noise \((f_c \sim \infty\) or \(\tau_N \sim 0\)). However, there is still a deviation of Eq. (2) from the experimental data for much lower \(f_c \sim (2kHz)\); a second approximation to the modification should be required in the future.

Moreover, Eq. (2) was checked for the dependence of slope \(b\) on ac-frequency \(f = \omega/(2\pi)\) as shown in Fig. 4. The sensitivity \(b(\omega) > 0\) and \(< 0\) is qualitatively explained by Eq. (2). For white noise \((f_c \sim \infty\) or \(\tau_N \sim 0\)), \(b_{mod}(\omega)\) [Eq. (2)] is also recovered to \(b(\omega)\) [Eq. (3)]. In particular, for \(\omega \ll \omega^*/2\pi = 2500Hz\), quantitatively good agreement between the modified curves and experimental data is found (see the inset of Fig. 4). However, for higher \(f > 1000Hz\), Eq. (2) shows a large deviation from the experimental data.
ECs,\textsuperscript{15,16} the saturation of \( b \) in experiment should be distinguished from the divergence in Eqs. (2) and (3); noise can suppress the divergence of \( V_c \) at \( \omega^* \) (see below for discussion).

Finally, we examined phase diagrams for ECs in the \( V_N-V \) plane (at \( \omega/2\pi = 30 \text{ Hz} \), considering the relationship, Eq. (4). Depending on \( \tau_N (\tau_N < \tau_0^* = 3.2 \times 10^{-5} \text{ s or } \tau_N > \tau_0^*) \), the phase diagrams are quite different from each other (Fig. 5). Similarly to the shift of the threshold for the primary instability (WD), the thresholds for other higher instabilities (FWD, GP, and DSM)\textsuperscript{1,16,17} also increase for \( \tau_N < \tau_0^* \) [Fig. 5(a)] and decrease for \( \tau_N > \tau_0^* \) [Fig. 5(c)] with increase of \( V_N \). Also, no shift of the thresholds is found for a critical colored noise (\( \tau_N \sim \tau_0^* \)) below \( V_N^* \sim 25 \text{ V} \) [Fig. 5(b)]. The most remarkable finding is the emergence of some unexpected patterns in Fig. 5(a). For higher \( V_N > V_N^* \sim 25 \text{ V} \), the typical, spatially-periodical patterns (WD, FWD, and GP) are replaced by a noise-dominated pattern (NDP).\textsuperscript{10} Such an NDP is evolved via target-like patterns (TP1 and TP2) with increase of \( V_N \); as a DSM (dynamic scattering mode) appears via the successive evolution of patterns (WD, FWD, and GP) with increase of \( V \). This is found only in white-like noise [\( \tau_N \ll \tau_0^* \), Fig 5(a)]. Although DSMs and NDPs may be indistinguishable in highly developed turbulences for much higher voltage (\( V \) and \( V_N \)), they are still distinguishable until our experimental limit (~200 V). For higher voltages, moreover, a DSM1-DSM2 transition\textsuperscript{17} was also observed (not seen in Fig. 5). In sufficiently colored noise (\( \tau_N \ll \tau_0^* \), Fig 5(c)), the noise (\( \tau_N < \tau_N < \tau_0 \)) plays a supplementary role in the destabilization effect on ac-driven ECs due to the Carr–Helfrich mechanism (\( \tau_N \ll \omega^{-1} \ll \tau_0 \)).\textsuperscript{15,16} It contributes to decreasing thresholds as a power effect (\( \tau_c \)).

As presented above, the previous theory [Eq. (1)] is sufficiently applicable to the impact of quasi-white noise on low \( \omega \)-driven EC thresholds (\( \omega \ll \omega^* \)), and our modified one [Eq. (2)] can expand the application range up to moderately colored noise. However, the modified one is still inapplicable for higher \( \omega \) frequencies and extremely strong colored noises. Probably, other factors such as high order terms of \( V_N^2 \) [e.g., \( \Omega(V_N^2) \)], an inertia term (or, viscous relaxation time \( \tau_{vis} \)), or a variation (and/or distribution) of wave vectors \( k \) may be required for further understanding.\textsuperscript{1}

It is worthwhile to mention that a completely different approach to the threshold shift by noise was reported in previous studies.\textsuperscript{1,18} It considered the well-known Fokker–Plank equation that leads us to nontrivial solutions at the threshold. Especially, in the work of Horsthemke et al.,\textsuperscript{1,18} the roles of noise such as both cases (\( b > 0 \) and \( b < 0 \)) were found: \( V_c^2 = V_{0c}^2 + \alpha(4\tau_N/\tau_d - 1)V_N^2 \). They considered the other characteristic time \( \tau_d \). For a real NLC (MBBA),...
however, the result was valid only for \( b < 0 \) (at \( \tau_N < \tau_\delta/4 \)), which is inapplicable to the ECs (for \( b > 0 \)). This was confirmed in the threshold shift of a Fredericks transition\(^{10}\) that is a homogeneous change (\( k = 0 \)) of the director orientation.\(^{16}\)

In conclusion, the relationships between the internal timescales of the EC system (\( \tau_\sigma \) and \( \omega^{-1} \)) and external one of noise (\( \tau_N \)) are crucial to understand the threshold shift and a variety of phase diagrams. The stabilization effect of noise to the EC threshold is switched to the destabilization effect with increasing \( \tau_N (> \tau_\sigma) \). The phase diagram of ECs can be clearly classified in white-like (\( \tau_N < \tau_\sigma^0 \)) and in colored (\( \tau_N > \tau_\sigma^0 \)) noises. For low \( \omega (\ll \omega^* \) small \( V_N (< V_N^0) \), and short \( \tau_N (< \tau_\sigma^0) \), the modified noise sensitivity \( b_{mod} \) [Eq. (2)] is quite effective for understanding colored noise-induced threshold shifts. More quantitative analysis certainly requires other factors such as an inertia term and wave vectors.\(^1\) Noise-induced phenomena out of equilibrium may be better understood in an inherent internal and external timescales as well as noise intensity. Generally, the impact of noise (or fluctuation) steeply increases with decrease of the size of systems. Therefore, controlling noise (not removing it) is very important in practical applications to various micro-systems in nanotechnology, biotechnology, neuroscience, etc., in which it is inevitable.\(^{19,20}\) Well-controlled, external colored noise can play a helpful role on such applications.

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